

Categorical Lévy Processes

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We generalize Franz' independence in tensor categories with inclusions from two morphisms (which represent generalized random variables) to arbitrary ordered families of morphisms. We will see that this only works consistently if the unit object is an initial object, in which case the inclusions can be defined starting from the tensor category alone. We define categorical Lévy processes on every tensor category with initial unit object and present a construction generalizing the reconstruction of a Lévy process from its convolution semigroup via the Daniell-Kolmogorov theorem.

1 Introduction

Suppose $(\mu_t)_{t \in \mathbb{R}_+}$ is a convolution semigroup of probability measures on the real line. Let us sketch, how to construct a Lévy process $X_t: \Omega \rightarrow \mathbb{R}$ with marginal distributions $\mathbb{P}_{X_t} = \mu_t$. First, for all finite subsets $J = \{t_1 < t_2 < \dots < t_n\} \subset \mathbb{R}_+$ define probability measures $\mu_J := \mu_{t_1} \otimes \mu_{t_2 - t_1} \otimes \dots \otimes \mu_{t_n - t_{n-1}}$ on \mathbb{R}^J . Then show that these are coherent in the sense that

$$\mu_I = \mu_J \circ (p_I^J)^{-1} \quad (1.1)$$

for all $I \subset J$ and $p_I^J: \mathbb{R}^J \rightarrow \mathbb{R}^I$ the canonical projection. In this situation the probability spaces $(\mathbb{R}^J, \mathcal{B}(\mathbb{R}^J), \mu_J)$ with the projections (p_I^J) form a *projective system*. Now, the Daniell-Kolmogorov theorem guarantees the existence of a *projective limit*, which is a probability

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space $(\Omega, \mathcal{F}, \mathbb{P})$ with projections $p_J: \Omega \rightarrow \mathbb{R}^J$ such that $\mu_J = \mathbb{P} \circ (p_J)^{-1}$. The random variable $X_t := p_{\{t\}}$ has distribution μ_t and the X_t have independent and stationary increments $X_s - X_t \sim \mu_{s-t}$.

A similar construction allows one to associate quantum Lévy processes with convolution semigroups of states on $*$ -bialgebras. The formulation of quantum probability is dual to that of classical probability, so inductive limits appear instead of projective limits. Due to the fact that there are different notions of independence in quantum probability on the one hand and the interactions between quantum probability and operator algebras on the other hand, there are many different theorems of the same kind (construction of Lévy processes for other notions of independence, see [Sch93, BGS05]) or similar kind (the construction of product systems from subproduct systems, cf. [BM10]). The main aim of these notes is to give a unified approach to these different situations. To this end, we introduce the language of tensor categories and the concept of comonoidal systems, which generalizes that of convolution semigroups.

With a similar goal in mind, Uwe Franz defined independence on *tensor categories with inclusions* [Fra06]. Whereas this works nicely for all questions concerning independence of two random variables, the approach runs into trouble when there are more random variables involved, which happens for example when one wants to study general properties of Lévy processes with respect to different notions of independence. Therefore, we consider tensor categories with initial unit object. We show that inclusions in the sense of Franz can always be defined in such a category and determine the necessary and sufficient conditions on given inclusions to be of the desired form. In this stronger setting Franz' notion of independence can be generalized to arbitrary ordered families of random variables. We define categorial Lévy processes on tensor categories with initial unit object and present a general construction reminding of the reconstruction of a Lévy process from its convolution semigroup via the Daniell-Kolmogorov theorem.

2 Basic Notions of Category Theory

The main point of this section is to fix notations and recall basic facts about inductive limits and tensor categories. We also give a list of those categories which appear as examples for the following sections.

We will freely use the language of categories, functors, and natural transformations, see for example [AHS04] for an exquisite treatment of the matter. For a category \mathcal{C} , we write $\mathbf{obj} \mathcal{C}$ for the class of objects of \mathcal{C} . Given two objects $A, B \in \mathbf{obj} \mathcal{C}$, we denote by $\mathbf{mor}(A, B)$ the set of all morphisms $f: A \rightarrow B$.

Equalities between morphisms will frequently be expressed in terms of commutative diagrams. A *diagram* is a directed graph with object-labeled vertices and morphism-labeled edges. We say that a diagram *commutes* if the composition of morphisms along any two directed paths with the same source and the same target vertex yield the same result. We

will usually not explicitly write the inverse of an isomorphism with an extra edge, but it shall be included when we say that the diagram commutes. If the morphism labelling an edge is a component α_A of a natural transformation α , and it is clear from the context (i.e. the source and the target object) which component we mean, we will drop the index to increase readability.

2.1 Inductive Limits

In category theory there are the general concepts of limits and colimits. Since in our applications only inductive limits play a role, we restrict to this special case. The general case can for example be found in the book of Adámek, Herrlich and Strecker [AHS04].

A preordered set I is called *directed* if any two Elements of I possess a common upper bound, that is if for all $\alpha, \beta \in I$ there exists $\gamma \in I$ with $\gamma \geq \alpha, \beta$.

2.1 Definition. Let \mathcal{C} be a category. An *inductive system* consists of

- a family of objects $(A_\alpha)_{\alpha \in I}$ indexed by a directed set I
- a family of morphisms $(f_\beta^\alpha: A_\alpha \rightarrow A_\beta)_{\alpha \leq \beta}$

such that

1. $f_\alpha^\alpha = \text{id}_{A_\alpha}$ for all $\alpha \in I$
2. $f_\gamma^\beta \circ f_\beta^\alpha = f_\gamma^\alpha$ for all $\alpha \leq \beta \leq \gamma$

An object \mathcal{A} together with morphisms $f^\alpha: A_\alpha \rightarrow \mathcal{A}$ for $\alpha \in I$ is called *inductive limit* of the inductive system $((A_\alpha)_{\alpha \in I}, (f_\beta^\alpha)_{\alpha \leq \beta})$ if

1. $f^\alpha = f^\beta \circ f_\beta^\alpha$ for all $\alpha \leq \beta$
2. whenever $g^\alpha = g^\beta \circ f_\beta^\alpha$ holds for a family of morphisms $g^\alpha: A_\alpha \rightarrow B$ to some common object B , there exists a unique morphism $g: \mathcal{A} \rightarrow B$ such that $g \circ f^\alpha = g^\alpha$ for all $\alpha \in I$. This is referred to as the *universal property* of the inductive limit.

If an inductive limit exists, it is essentially unique. More precisely, if $(\mathcal{A}, (f^\alpha)_{\alpha \in I})$ and $(\mathcal{B}, (g^\alpha)_{\alpha \in I})$ are two inductive limits of the same inductive system $(A_\alpha)_{\alpha \in I}$, then the uniquely determined morphisms $f: \mathcal{A} \rightarrow \mathcal{B}$ with $f \circ f^\alpha = g^\alpha$ and $g: \mathcal{B} \rightarrow \mathcal{A}$ with $g \circ g^\alpha = f^\alpha$ are mutually inverse isomorphisms.

In general, inductive limits may or may not exist. We call a category in which all inductive systems have inductive limits *inductively complete*. See [AHS04, Chapter 12] for general arguments showing that many categories we consider even fulfill the stronger property of *cocompleteness*.

A subset J of a directed set I is called *cofinal* if for every $\alpha \in I$ there exists a $\beta \in J$ with $\beta \geq \alpha$.

2.2 Example. For any fixed $\alpha_0 \in I$ the set $\{\beta \mid \beta \geq \alpha_0\}$ is cofinal. Indeed, since I is directed, there is a $\beta \geq \alpha, \alpha_0$ for all $\alpha \in I$.

Clearly, if $((A_\alpha)_{\alpha \in I}, (f_\beta^\alpha)_{\alpha \leq \beta, \alpha, \beta \in I})$ is an inductive system and $J \subset I$ cofinal, then also $((A_\alpha)_{\alpha \in J}, (f_\beta^\alpha)_{\alpha \leq \beta, \alpha, \beta \in J})$ is an inductive system. It is known that the inductive limits are canonically isomorphic if they exist. We will need the following generalization of this. Let $(A_\alpha)_{\alpha \in I}$ be an inductive system with inductive limit $(\mathcal{A}, (f^\alpha)_{\alpha \in I})$, K a directed set and $J_k \subset I$ for each $k \in K$ such that

- J_k is directed for all $k \in K$
- $J_k \subset J_{k'}$ for all $k \leq k'$
- $J = \bigcup_{k \in K} J_k$ cofinal in I .

Suppose the inductive systems $(A_\alpha)_{\alpha \in J_k}$ have inductive limits $(\mathcal{A}_k, (f_{(k)}^\alpha)_{\alpha \in J_k})$. It holds that $f^\alpha = f^\beta \circ f_\beta^\alpha$ for all $\alpha \leq \beta \in J_k$, since $J_k \subset I$. Similarly, for $k \leq k'$ it holds that $f_{(k')}^\alpha = f_{(k')}^\beta \circ f_\beta^\alpha$ for all $\alpha \leq \beta \in J_k$, since $J_k \subset J_{k'}$. By the universal property of the inductive limit \mathcal{A}_k there are unique morphisms $f_{k'}^k: \mathcal{A}_k \rightarrow \mathcal{A}_{k'}$ for $k \leq k'$ and $f^k: \mathcal{A}_k \rightarrow \mathcal{A}$ such that the diagrams

$$\begin{array}{ccc} A_\alpha & \xrightarrow{f^\alpha} & \mathcal{A} \\ \downarrow f_{(k)}^\alpha & \nearrow f^k & \\ \mathcal{A}_k & & \end{array} \quad \begin{array}{ccc} A_\alpha & \xrightarrow{f_{(k')}^\alpha} & \mathcal{A}_{k'} \\ \downarrow f_{(k)}^\alpha & \nearrow f_{k'}^k & \\ \mathcal{A}_k & & \end{array}$$

commute for all $\alpha \in J_k$.

2.3 Proposition. In the described situation $((\mathcal{A}_k)_{k \in K}, (f_{k'}^k)_{k \leq k'})$ is an inductive system with inductive limit $(\mathcal{A}, (f^k)_{k \in K})$.

Proof. The diagrams

$$\begin{array}{ccc} A_\alpha & \xrightarrow{f^\alpha} & \mathcal{A} \\ \downarrow f_{(k)}^\alpha & \searrow f_{(k')}^\alpha & \uparrow f^{k'} \\ \mathcal{A}_k & \xrightarrow{f_{k'}^k} & \mathcal{A}_{k'} \end{array} \quad \begin{array}{ccc} A_\alpha & \xrightarrow{f_{(k'')}^\alpha} & \mathcal{A}_{k''} \\ \downarrow f_{(k)}^\alpha & \searrow f_{(k')}^\alpha & \uparrow f_{k''}^{k'} \\ \mathcal{A}_k & \xrightarrow{f_{k'}^k} & \mathcal{A}_{k'} \end{array}$$

commute, which implies $f^k = f^{k'} \circ f_{k'}^k$ and $f_{k''}^k = f_{k''}^{k'} \circ f_{k'}^k$ for all $k \leq k' \leq k''$. Now suppose there are $g^k: \mathcal{A}_k \rightarrow \mathcal{B}$ with $g^k = g^{k'} \circ f_{k'}^k$ for all $k \leq k'$. We put $g^\alpha := g^k \circ f_{(k)}^\beta \circ f_\beta^\alpha$ for $\beta \in J_k$, $\alpha \leq \beta$. Since $J = \bigcup_{k \in K} J_k$ is cofinal in I , we can find such k and β for every α . One can check that the g^α do not depend on the choice and fulfill $g^\alpha = g^{\alpha'} \circ f_{\alpha'}^\alpha$ for all $\alpha \leq \alpha'$. This

yields a morphism $g: \mathcal{A} \rightarrow \mathcal{B}$ which makes

$$\begin{array}{ccc}
 \mathcal{A}_k & \xrightarrow{g^k} & \mathcal{B} \\
 \uparrow f_{(k)}^\beta & \searrow f^k & \uparrow g \\
 A_\beta & \xrightarrow{f^\beta} & \mathcal{A} \\
 \uparrow f_\beta^\alpha & \nearrow f^\alpha & \\
 A_\alpha & &
 \end{array}$$

commute. On the other hand, any morphism which makes the upper right triangle commute, automatically makes the whole diagram commute and will therefore equal g . \square

2.4 Corollary. *Let $(A_\alpha)_{\alpha \in I}$ be an inductive system with inductive limit $(\mathcal{A}, (f^\alpha)_{\alpha \in I})$, $J \subset I$ cofinal. Then $(A_\alpha)_{\alpha \in J}$ is an inductive system with inductive limit $(\mathcal{A}, (f^\alpha)_{\alpha \in J})$.*

Proof. This is a special case of the previous theorem with $|K| = 1$, since the inductive system over the one point set K does not add anything. \square

2.2 Tensor Categories

A *tensor category* is a category \mathcal{C} together with a bifunctor $\boxtimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ which

- is associative under a natural isomorphism with components

$$\alpha_{A,B,C}: A \boxtimes (B \boxtimes C) \xrightarrow{\cong} (A \boxtimes B) \boxtimes C$$

called *associativity constraint*,

- has a unit object $E \in \mathbf{obj}(\mathcal{C})$ acting as left and right identity under natural isomorphisms with components

$$l_A: E \boxtimes A \xrightarrow{\cong} A, \quad r_A: A \boxtimes E \xrightarrow{\cong} A$$

called *left unit constraint* and *right unit constraint* respectively

such that the pentagon and triangle identities hold [ML98]. If the natural transformations α, l and r are all identities, we say the tensor category is *strict*.

It can be shown that the pentagon and triangle identities imply commutativity of all diagrams which only contain α, l and r [ML98, VII.2]. This is called MacLane's *coherence* theorem. Even for non-strict tensor categories, we will frequently suppress the associativity and unit constraints in the notation and write $(\mathcal{C}, \boxtimes, E)$, or even (\mathcal{C}, \boxtimes) or \mathcal{C} . In the examples we treat, α, l and r are always canonical.

3 Independence in Tensor Categories

3.1 Categorical Independence with respect to inclusions

In order to unify the different notions of independence in quantum probability, Franz came up with a definition of independence in a tensor-categorical framework [Fra06, Section 3]. Let $\mathcal{P}_i: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ for $i \in \{1, 2\}$ denote the projection functor onto the first or second component respectively.

3.1 Definition. Let (\mathcal{C}, \boxtimes) be a tensor category. A natural transformation $\iota^1: \mathcal{P}_1 \Rightarrow \boxtimes$ is called *left inclusion* and a natural transformations $\iota^2: \mathcal{P}_2 \Rightarrow \boxtimes$ is called *right inclusion*. A tensor category together with a right and a left inclusion is referred to as *tensor category with inclusions*.

In more detail, inclusions are two collections of morphisms $\iota_{B_1, B_2}^i: B_i \rightarrow B_1 \boxtimes B_2$ for $B_1, B_2 \in \mathcal{C}$, $i \in \{1, 2\}$ such that

$$\begin{array}{ccccc} A_1 & \xrightarrow{\iota^1} & A_1 \boxtimes A_2 & \xleftarrow{\iota^2} & A_2 \\ f_1 \downarrow & & \downarrow f_1 \boxtimes f_2 & & \downarrow f_2 \\ B_1 & \xrightarrow{\iota^1} & B_1 \boxtimes B_2 & \xleftarrow{\iota^2} & B_2 \end{array}$$

commutes for all $f_i: A_i \rightarrow B_i$, $i \in \{1, 2\}$.

3.2 Definition. Let $(\mathcal{C}, \boxtimes, \iota^1, \iota^2)$ be a tensor category with inclusions. Two morphisms $j_1, j_2: B_i \rightarrow A$ are *independent* if there exists a morphism $h: B_1 \boxtimes B_2 \rightarrow A$ such that the diagram

$$\begin{array}{ccccc} & & A & & \\ & j_1 \nearrow & \uparrow h & \nwarrow j_2 & \\ B_1 & \xrightarrow{\iota^1} & B_1 \boxtimes B_2 & \xleftarrow{\iota^2} & B_2 \end{array}$$

commutes. Such a morphism h is called *independence morphism* for j_1 and j_2 .

Let us first consider the case where the tensor product $B_1 \boxtimes B_2$ coincides with the *coproduct* $B_1 \sqcup B_2$ in the category; see [Fra06] or [McL92]. By definition of a coproduct, any pair of morphisms $j_i: B_i \rightarrow A$ to a common target A will be independent with the unique independence morphism $h = j_1 \sqcup j_2$. Coproducts exist in many categories, for example the direct sum in the category of vector spaces with linear maps, or the free product in the category of algebras with algebra homomorphisms. In order to have a nontrivial notion of independence, one should either use a different tensor product, or restrict the class of morphisms. In Section 5 we will see that many notions of independence used in mathematics are indeed special cases of categorical independence, in particular this holds for linear independence, orthogonality, stochastic independence, Bose or tensor independence, Fermi independence, and all notions

of noncommutative stochastic independence which are induced by universal products, like freeness, Boolean independence and monotone independence.

In most examples the independence morphism h will be uniquely determined if it exists. The next example shows that this is not the case in general, which is why we will not assume uniqueness in the development of the general theory.

3.3 Example. Consider the category \mathbf{vcc} with tensor product

$$V_1 \odot V_2 := V_1 \oplus V_2 \oplus V_1 \otimes V_2.$$

and the canonical inclusions $V_i \hookrightarrow V_1 \odot V_2$ which identify V_i with the summand V_i in $V_1 \oplus V_2$. Any two linear maps $f_i: V_i \rightarrow W$ are independent, but the independence morphism is not uniquely determined. Indeed, for an arbitrary linear map $f: V_1 \otimes V_2 \rightarrow W$, the linear map $h = f_1 + f_2 + f$ is an independence morphism for f_1 and f_2 .

3.2 Compatible Inclusions

We already defined what it means for two morphisms j_1, j_2 in a tensor category with inclusions to be independent. But if we want to consider notions of independence for more than two morphisms, we need to require certain compatibility conditions between the inclusions and the structure of the tensor category. In particular, for dealing with categorial Lévy processes (as we will do in Section 4.6) it seems to be necessary. This has first been observed in [Ger15] and the conditions do not appear in [Fra06].

3.4 Definition. Inclusions ι^1, ι^2 are called *compatible with the unit constraints* if the diagram

$$\begin{array}{ccccc} E \boxtimes A & \xleftarrow{\iota_2} & A & \xrightarrow{\iota_1} & A \boxtimes E \\ & \searrow l_A & \downarrow \text{id}_A & \swarrow r_A & \\ & & A & & \end{array} \quad (3.1)$$

commutes for all objects $A \in \mathcal{C}$.

3.5 Theorem. *Let \mathcal{C} be a tensor category.*

- (a) *If ι^1, ι^2 are inclusions which are compatible with the unit constraints, then the unit object E is initial, i.e. there is a unique morphism $1_A: E \rightarrow A$ for every object $A \in \mathcal{C}$. Furthermore*

$$\iota_{A,B}^1 = (\text{id} \boxtimes 1_B) \circ r_A^{-1}, \quad \iota_{A,B}^2 = (1_A \boxtimes \text{id}) \circ l_B^{-1} \quad (3.2)$$

holds for all objects $A, B \in \mathcal{C}$.

- (b) Suppose that the unit object E is an initial object. Then (3.2) read as a definition yields inclusions ι^1, ι^2 which are compatible with the unit constraints.

Proof. (a): The inclusions can easily be used to define the morphism $1_A := l_A \circ \iota_{E,A}^1$ from E to an arbitrary object A . To prove that E is initial, it remains to show that any morphism $f : E \rightarrow A$ coincides with 1_A . Naturality of l yields $f \circ l_E = l_A \circ (\text{id} \boxtimes f)$. By coherence $r_E = l_E$ and from the compatibility with the unit constraints we know that $\iota_{E,E}^1 : E \rightarrow E \boxtimes E$ and $r_E = l_E : E \boxtimes E \rightarrow E$ are mutually inverse isomorphisms. Thus, solving for f yields $f = l_A \circ (\text{id} \boxtimes f) \circ \iota_{E,E}^1$. Naturality of ι^1 implies that $(\text{id} \boxtimes f) \circ \iota_{E,E}^1 = \iota_{E,A}^1$. Therefore $f = l_A \circ \iota_{E,A}^1 = 1_A$, which shows that E is initial. In the diagram

$$\begin{array}{ccccc}
 A & & \xrightarrow{\iota^1} & & A \boxtimes B \\
 & \searrow \iota^1 & & \nearrow \text{id} \boxtimes r & \\
 & & A \boxtimes (B \boxtimes E) & & \\
 & \nearrow r^{-1} = \iota^1 & \uparrow \text{id} \boxtimes \iota^2 & \searrow \text{id} \boxtimes 1_B & \\
 & & A \boxtimes E & &
 \end{array}$$

the upper triangle and the lower left triangle commute due to the naturality of ι^1 . Because E is initial, it holds that $1_B := r_A \circ \iota_{A,E}^2$, so the lower right triangle also commutes. So the whole diagram commutes and the outside triangle represents the first equation of (3.2). The second one follows analogously.

(b): It follows from uniqueness of morphisms from E to B that $1_B = 1_A \circ f$ for all $f : A \rightarrow B$, or put as a diagram,

$$\begin{array}{ccc}
 E & \xrightarrow{1_A} & A \\
 \text{id}_E \downarrow & & \downarrow f \\
 E & \xrightarrow{1_B} & B
 \end{array}$$

commutes. Thus, we can interpret the collection of all $1_A, A \in \mathcal{C}$ as a natural transformation $1 : E \Rightarrow \text{id}_{\mathcal{C}}$; here E stands for the constant functor $E : \mathcal{C} \rightarrow \mathcal{C}$ with $E(A) = E$ for all objects A and $E(f) = \text{id}_E$ for all morphisms f . With this, naturality of ι^1, ι^2 is easy to check. We also have that $1_E = \text{id}_E$. Hence, $r_A \circ \iota_{A,E}^2 = r_A \circ (\text{id}_A \boxtimes 1_E) \circ r_A^{-1} = r_A r_A^{-1} = \text{id}_A$ for all objects A . Analogously, $l_A \circ \iota_{E,A}^1 = \text{id}_A$. \square

3.6 Observation. MacLanes's coherence theorem extends to all diagrams build up from the natural transformations α, l, r and 1 . The basic idea is the following. Suppose f_1 and f_2 are parallel morphisms build up from the natural transformations α, l, r and 1 . We can commute instances of 1 with instances of α, l and r using naturality, for example

$1_A \circ r_E = r_A \circ (1_A \boxtimes \text{id}_E)$. This allows us to subsequently move all instances of 1 to the right of all instances of α , l and r . Thus, we can factorize f_i as $g_i \circ h_i$ with g_i build up from α , l and r and h_i from 1 only. It is not hard to see that $h_1 = h_2$. In particular g_1 and g_2 are parallel, so they are equal by MacLanes coherence theorem. Altogether we see that indeed $f_1 = g_1 \circ h_1 = g_2 \circ h_2 = f_2$.

For a formal proof there are some subtleties to consider, mainly caused by the problem that in a concrete tensor category it can happen that tensor products of different factors can yield the same object. One has to be very careful what is meant by "build up from natural transformations", because one cannot conclude, just by looking at the object, which instance of a natural transformation is the right one to apply. But all of these subtleties are the same in MacLanes original coherence theorem, so we refer the reader to the discussion on this in the literature.

Particular coherence conditions which involve the associativity constraints and the inclusions and which we will need later on are commutativity of the following diagrams.

$$\begin{array}{ccc}
 A \boxtimes C & \xrightarrow{\iota^1 \boxtimes \text{id}_C} & (A \boxtimes B) \boxtimes C \\
 \text{id}_A \boxtimes \iota^2 \downarrow & & \nearrow \alpha_{A,B,C} \\
 A \boxtimes (B \boxtimes C) & &
 \end{array} \tag{3.3}$$

$$\begin{array}{ccc}
 A \boxtimes B \xleftarrow{\iota^2} B \xrightarrow{\iota^1} B \boxtimes C & & C \xrightarrow{\iota^2} B \boxtimes C \\
 \downarrow \iota^1 & & \downarrow \iota^2 \\
 (A \boxtimes B) \boxtimes C \xrightarrow{\alpha} A \boxtimes (B \boxtimes C) & & (A \boxtimes B) \boxtimes C \xrightarrow{\alpha} A \boxtimes (B \boxtimes C)
 \end{array}$$

3.3 General Categorical Independence

Let \mathcal{C} be a tensor category such that the unit object E is an initial object. By Observation 3.6 there are unique morphisms $\iota_{A_1, \dots, A_n}^{i_1, \dots, i_k; n}: A_{i_1} \boxtimes \dots \boxtimes A_{i_k} \rightarrow A_1 \boxtimes \dots \boxtimes A_n$ built up from l^{-1} , r^{-1} and 1. These constitute natural transformations $\iota^{i_1, \dots, i_k; n}$, again referred to as inclusions. The formerly considered inclusions $\iota^1 = \iota^{1;2}$ and $\iota^2 = \iota^{2;2}$ are special cases.

3.7 Definition. Let B_1, \dots, B_n, A be objects of \mathcal{C} and $f_i: B_i \rightarrow A$ morphisms. Then f_1, \dots, f_n are called *independent* if there exists a morphism $h: B_1 \boxtimes \dots \boxtimes B_n \rightarrow A$ such that the diagrams

$$\begin{array}{ccc}
 & & A \\
 & \nearrow f_i & \uparrow h \\
 B_i & \xrightarrow{\iota^{i;n}} & B_1 \boxtimes \dots \boxtimes B_n
 \end{array}$$

commute for all $i \in \{1, \dots, n\}$.

We conclude with the analogues of two basic results about stochastic independence: Subfamilies of independent families of random variables are independent and functions of independent random variables are independent.

3.8 Theorem. *Let \mathcal{C} be a tensor category with initial unit object. Furthermore let $f_1, \dots, f_n, f_i: B_i \rightarrow A$, be independent with independence morphism $h: B_1 \boxtimes \dots \boxtimes B_n \rightarrow A$. Then the following holds.*

- (a) *For all $1 \leq i_1 < \dots < i_k \leq n$ the morphisms f_{i_1}, \dots, f_{i_k} are independent with independence morphism $h \circ \iota^{i_1, \dots, i_k; n}$.*
- (b) *Let $j_1, \dots, j_1, j_i: C_i \rightarrow B_i$, be morphisms and put $g_i := f_i \circ j_i: C_i \rightarrow A$. Then g_1, \dots, g_n are independent with independence morphism $h \circ (j_1 \boxtimes \dots \boxtimes j_n)$.*

Proof. (a): In the diagram

$$\begin{array}{ccccc}
 & & & A & \\
 & & f_{i_j} \nearrow & \uparrow h & \\
 B_{i_j} & \xrightarrow{\iota^{i_j; n}} & B_1 \boxtimes \dots \boxtimes B_n & & \\
 & \searrow \iota^{j; k} & \uparrow \iota^{i_1, \dots, i_k; n} & & \\
 & & B_{i_1} \boxtimes \dots \boxtimes B_{i_k} & &
 \end{array}$$

the upper half commutes by independence of f_1, \dots, f_n and the lower half by coherence (see Observation 3.6). Commutativity for every $j = 1, \dots, k$ proves the assertion.

(b): In the diagram

$$\begin{array}{ccccc}
 C_i & \xrightarrow{j_i} & B_i & & \\
 \downarrow \iota^{i; n} & & \downarrow \iota^{i; n} & \searrow f_i & \\
 C_1 \boxtimes \dots \boxtimes C_n & \xrightarrow{j_1 \boxtimes \dots \boxtimes j_n} & B_1 \boxtimes \dots \boxtimes B_n & \xrightarrow{h} & A
 \end{array}$$

the left hand side square commutes because $\iota^{i; n}$ is a natural transformation and the right hand side triangle commutes by independence of f_1, \dots, f_n . So we get $g_i = f_i \circ j_i = h \circ (j_1 \boxtimes \dots \boxtimes j_n) \circ \iota_{C_1, \dots, C_n}^{i; n}$ for all i , which was the assertion. \square

We shortly discuss independence for infinite families. Let $(f_i: B_i \rightarrow A)_{i \in I}$ be a family of morphisms indexed by a totally ordered index set (I, \leq) . The set $P_{fin}(I) = \{\{i_1 < \dots < i_n\} \mid n \in \mathbb{N}, i_1, \dots, i_n \in I\}$ of all finite subsets of I is a directed set with respect to inclusion. For $\mathbf{i} = \{i_1 < \dots < i_n\} \in P_{fin}(I)$ $B_{\mathbf{i}} := B_{i_1} \boxtimes \dots \boxtimes B_{i_n}$ and $f_{\mathbf{i}} := f_{i_1} \boxtimes \dots \boxtimes f_{i_n}: B_{\mathbf{i}} \rightarrow A$. Given two finite subsets $\mathbf{i}, \mathbf{j} \in P_{fin}(I)$ with $\mathbf{i} \subset \mathbf{j}$, we put $\iota_{\mathbf{i}}^{\mathbf{j}}: B_{\mathbf{i}} \rightarrow B_{\mathbf{j}}$ as the unique morphism described

in Observation 3.6. By the same observation we know that $\iota_j^k \iota_i^j = \iota_i^k$ for all $\mathbf{i} \subset \mathbf{j} \subset \mathbf{k}$. In other words, $((B_{\mathbf{i}})_{\mathbf{i} \in P_{fin}(I)}, (f_{\mathbf{i}}^j)_{\mathbf{i} \subset \mathbf{j}})$ is an inductive system. We say that the $(f_i)_{i \in I}$ are independent if there is an inductive limit $(B_I, (\iota_i: B_{\mathbf{i}} \rightarrow B_I)_{\mathbf{i} \in P_{fin}(I)})$ of $((B_{\mathbf{i}})_{\mathbf{i} \in P_{fin}(I)}, (\iota_i^j)_{\mathbf{i} \subset \mathbf{j}})$ and a morphism $h: B_I \rightarrow A$ such that $h \circ \iota^{\{i\}} = f_i$ for all $i \in I$.

4 Lévy Processes in Tensor Categories

In this section we define comonoidal systems and present two important inductive limit constructions which have been considered for many different examples. In the context of Hilbert modules these constructions have been described by Bhat and Skeide [BS00], who also refer to them as first and second inductive limit.

4.1 Cotensor Functors

Given tensor categories (\mathcal{C}, \boxtimes) and $(\mathcal{C}', \boxtimes')$ with unit objects, associativity and unit constraints E, α, l, r and E', α', l', r' respectively, a *cotensor functor* is a triple $(\mathcal{F}, \delta, \Delta)$ consisting of

- a functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{C}'$
- a morphism $\delta: \mathcal{F}(E) \rightarrow E'$
- a natural transformation $\Delta: \mathcal{F}(\cdot \boxtimes \cdot) \Rightarrow \mathcal{F}(\cdot) \boxtimes' \mathcal{F}(\cdot)$

such that the diagrams

$$\begin{array}{ccc}
 \mathcal{F}(A \boxtimes (B \boxtimes C)) & \xrightarrow{\mathcal{F}(\alpha_{A,B,C})} & \mathcal{F}((A \boxtimes B) \boxtimes C) \\
 \Delta_{A,B \boxtimes C} \downarrow & & \downarrow \Delta_{A \boxtimes B, C} \\
 \mathcal{F}(A) \boxtimes' \mathcal{F}(B \boxtimes C) & & \mathcal{F}(A \boxtimes B) \boxtimes' \mathcal{F}(C) \\
 \text{id}_{\mathcal{F}(A)} \boxtimes' \Delta_{B,C} \downarrow & & \downarrow \Delta_{A,B} \boxtimes' \text{id}_{\mathcal{F}(C)} \\
 \mathcal{F}(A) \boxtimes' (\mathcal{F}(B) \boxtimes' \mathcal{F}(C)) & \xrightarrow{\alpha'_{\mathcal{F}(A), \mathcal{F}(B), \mathcal{F}(C)}} & (\mathcal{F}(A) \boxtimes' \mathcal{F}(B)) \boxtimes' \mathcal{F}(C)
 \end{array} \tag{4.1}$$

$$\begin{array}{ccc}
 \mathcal{F}(B \boxtimes E) & \xrightarrow{\Delta_{B,E}} & \mathcal{F}(B) \boxtimes' \mathcal{F}(E) \\
 \mathcal{F}(r_B) \downarrow & & \downarrow \text{id}_{\mathcal{F}(B)} \boxtimes' \delta \\
 \mathcal{F}(B) & \xleftarrow{r'_{\mathcal{F}(B)}} & \mathcal{F}(B) \boxtimes' E'
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{F}(E \boxtimes B) & \xrightarrow{\Delta_{E,B}} & \mathcal{F}(E) \boxtimes' \mathcal{F}(B) \\
 \mathcal{F}(l_B) \downarrow & & \downarrow \delta \boxtimes' \text{id}_{\mathcal{F}(B)} \\
 \mathcal{F}(B) & \xleftarrow{l'_{\mathcal{F}(B)}} & E' \boxtimes' \mathcal{F}(B)
 \end{array} \tag{4.2}$$

commute for all $A, B, C \in \mathbf{obj}(\mathcal{C})$. A cotensor functor is called *strong* if Δ is a natural isomorphism and δ is an isomorphism.

4.1 Theorem. Let $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{C}'$ and $\mathcal{F}': \mathcal{C}' \rightarrow \mathcal{C}''$ be cotensor functors with coproduct morphisms $\Delta_{A,B}, \Delta'_{A',B'}$ and counit morphisms δ, δ' . Then $\mathcal{F}' \circ \mathcal{F}$ is a cotensor functor with coproduct morphisms $\Delta'_{\mathcal{F}(A), \mathcal{F}(B)} \circ \mathcal{F}'(\Delta_{A,B})$ and counit morphism $\delta' \circ \mathcal{F}'(\delta)$.

This is well known and can be shown by writing down the involved diagrams and check that they commute; see [Lac15] for an explicit proof.

Similarly, a *tensor functor* is a functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{C}'$ together with a natural transformation $\mu: \mathcal{F}(\cdot) \boxtimes \mathcal{F}(\cdot) \Rightarrow \mathcal{F}(\cdot \boxtimes \cdot)$ and a morphism $1: E' \rightarrow \mathcal{F}(E)$ such that the diagrams one obtains from (4.1) and (4.2) by reversing the arrows and replacing Δ and δ with μ and 1 commute.

4.2 Comonoidal Systems

A *monoid* is a semigroup with a unit element. We identify a monoid \mathbb{S} with the strict tensor category whose objects are the elements of \mathbb{S} with only the identity morphisms and the tensor product given by the multiplication of \mathbb{S} .

4.2 Definition. Let \mathbb{S} be a monoid and (\mathcal{C}, \boxtimes) a tensor category. A *monoidal system* over \mathbb{S} in \mathcal{C} is a tensor functor from \mathbb{S} to \mathcal{C} . A *comonoidal system* over \mathbb{S} in \mathcal{C} is a cotensor functor from \mathbb{S} to \mathcal{C} . A comonoidal system is called *full* if the cotensor functor is strong. A monoidal system (respectively comonoidal system) over the trivial monoid $\{e\}$ is simply called a *monoid in \mathcal{C}* (respectively *comonoid in \mathcal{C}*).

Since there are only identity morphisms in \mathbb{S} , any functor defined on \mathbb{S} acts trivially on morphisms, so it is determined by the object assignment and can be identified with the family $(A_s)_{s \in \mathbb{S}}$ where A_s denotes the value of the functor at $s \in \mathbb{S}$. Thus, a monoidal system over \mathbb{S} in \mathcal{C} is the same as a family of objects $(A_s)_{s \in \mathbb{S}}$ together with *product morphisms* $\mu_{s,t}: A_s \boxtimes A_t \rightarrow A_{st}$ and a *unit morphism* $u: E \rightarrow A_e$ such that the natural associativity and unit properties

$$\begin{array}{ccc} A_r \boxtimes A_s \boxtimes A_t & \xrightarrow{\mu_{r,s} \boxtimes \text{id}_t} & A_{rs} \boxtimes A_t \\ \downarrow \text{id}_r \boxtimes \mu_{s,t} & & \downarrow \mu_{rs,t} \\ A_r \boxtimes A_{st} & \xrightarrow{\mu_{r,st}} & A_{rst} \end{array} \quad \begin{array}{ccccc} E \boxtimes A_s & \xleftarrow{l_{A_s}^{-1}} & A_s & \xrightarrow{r_{A_s}^{-1}} & A_s \boxtimes E \\ \downarrow u \boxtimes \text{id} & & \downarrow \text{id} & & \downarrow \text{id} \boxtimes u \\ A_e \boxtimes A_s & \xrightarrow{\mu_{e,s}} & A_s & \xleftarrow{\mu_{s,e}} & A_s \boxtimes A_e \end{array}$$

are fulfilled. In particular, a monoid in **set** is just a usual monoid. Similarly, a comonoidal system over \mathbb{S} in \mathcal{C} is a family of objects $(A_s)_{s \in \mathbb{S}}$ together with *coproduct morphisms* $\Delta_{s,t}: A_{st} \rightarrow A_s \boxtimes A_t$ and a *counit morphism* $\delta: A_e \rightarrow E$ such that coassociativity and the counit properties

$$\begin{array}{ccc} A_{rst} & \xrightarrow{\Delta_{rs,t}} & A_{rs} \boxtimes A_t \\ \downarrow \Delta_{r,st} & & \downarrow \Delta_{r,s} \boxtimes \text{id}_t \\ A_r \boxtimes A_{st} & \xrightarrow{\text{id}_r \boxtimes \Delta_{s,t}} & A_r \boxtimes A_s \boxtimes A_t \end{array} \quad \begin{array}{ccccc} A_e \boxtimes A_s & \xleftarrow{\Delta_{e,s}} & A_s & \xrightarrow{\Delta_{s,e}} & A_s \boxtimes A_e \\ \downarrow \delta \boxtimes \text{id} & & \downarrow \text{id} & & \downarrow \text{id} \boxtimes \delta \\ E \boxtimes A_s & \xrightarrow{l_{A_s}} & A_s & \xleftarrow{r_{A_s}} & A_s \boxtimes E \end{array}$$

hold. The composition of two cotensor functors is again a cotensor functor in the sense of Theorem 4.1. This immediately implies that a cotensor functor $(\mathcal{F}, \mathcal{D}, d)$ maps a comonoidal system $(A_s)_{s \in \mathbb{S}}$ with coproduct morphisms $\Delta_{s,t}$ and counit morphism δ to a comonoidal system $(\mathcal{F}(A_s))_{s \in \mathbb{S}}$ with coproduct morphisms $\mathcal{D}_{A_s, A_t} \circ \mathcal{F}(\Delta_{s,t})$ and counit morphism $d \circ \mathcal{F}(\delta)$. The analogous statements hold for monoidal systems.

In the following, we concentrate on comonoidal systems, because they are more important in the subsequent chapters. Most results have an obvious corresponding result for monoidal systems.

4.3 Theorem. *Let $\mathbb{U} \subset \mathbb{S}$ be a submonoid. If $(A_s)_{s \in \mathbb{S}}$ is a comonoidal system with coproduct morphisms $(\Delta_{s,t})_{s,t \in \mathbb{S}}$ and counit morphism δ , then $(A_s)_{s \in \mathbb{U}}$ is a comonoidal system with coproduct morphisms $(\Delta_{s,t})_{s,t \in \mathbb{U}}$ and counit morphism δ .*

Proof. The inclusion $\mathbb{U} \hookrightarrow \mathbb{S}$ is a monoid homomorphism, hence it is a cotensor functor with respect to the identity natural transformation and identity morphism. The theorem now follows from Theorem 4.1. \square

4.4 Definition. Let $(A_t)_{t \in \mathbb{S}}$ be a comonoidal system in \mathcal{C} . A *categorical Lévy-process* on $(A_t)_{t \in \mathbb{S}}$ is a collection of morphisms $j_{s,t}: A_{t-s} \rightarrow B$ for $s \leq t$ to some common object $B \in \mathcal{C}$ such that

1. $j_{t,t} = 1_B \circ \delta$
2. $j_{s_1, t_1}, \dots, j_{s_n, t_n}$ are independent if $s_1 \leq t_1 \leq s_2 \leq \dots \leq s_n \leq t_n$
3. $j_{r,s,t} \circ \Delta_{s-r, t-s} = j_{r,s}$ for some independence morphism $j_{r,s,t}$ of $j_{r,s}$ and $j_{s,t}$.

In the following we exhibit general conditions on the structure of the index set \mathbb{S} and the category \mathcal{C} which ensure that comonoidal systems embed into full comonoidal systems (Section 4.4) and that every comonoidal system allows for the construction of a canonical categorical Lévy process on it.

4.3 Unique Factorization Monoids

A monoid \mathbb{S} is called *cancellative* if $ab = ac$ implies $b = c$ and $ba = ca$ implies $b = c$ for all $a \in \mathbb{S}$. Note that left invertibility, right invertibility and invertibility are all equivalent for elements of a cancellative monoid \mathbb{S} . Indeed, suppose $ab = e$ with $e \in \mathbb{S}$ the unit element. This implies $baba = bea = bae$. Since \mathbb{S} is cancellative it follows that $ba = e$ and hence $a = b^{-1}$.

Denote by \mathbb{S}^* the set of all tuples (s_1, \dots, s_n) over \mathbb{S} of arbitrary length $n \in \mathbb{N}_0$. The concatenation of tuples is written in this section as

$$(s_1, \dots, s_n) \smile (t_1, \dots, t_m) := (s_1, \dots, s_n, t_1, \dots, t_m)$$

to clearly distinguish it from the monoid multiplication.

A tuple $(s_1, \dots, s_n) \in \mathbb{S}^*$ is called a *factorization* of $t \in \mathbb{S}$ if $t = s_1 \cdots s_n$ with $s_i \in \mathbb{S} \setminus \{e\}$. The set of all factorizations of $t \in \mathbb{S}$ is denoted by \mathbb{F}_t . A factorization $\sigma \in \mathbb{F}_t$ is said to be a *refinement* of a factorization $(t_1, \dots, t_n) \in \mathbb{F}_t$ if $\sigma = \tau_1 \cup \dots \cup \tau_n$ for some $\tau_k \in \mathbb{F}_{t_k}$. We write $\sigma \geq \tau$ if σ is a refinement of τ . This defines a partial order on \mathbb{F}_t .

We say that \mathbb{S} is *conical* if the only invertible element of \mathbb{S} is its unit element e . If \mathbb{S} is conical, the empty tuple $()$ is the unique factorization of e .

4.5 Proposition. *Let \mathbb{S} be a cancellative, conical monoid. If $\tau_1 \cup \dots \cup \tau_n = \tau'_1 \cup \dots \cup \tau'_n$ with $\tau_k, \tau'_k \in \mathbb{F}_{t_k}$, then $\tau_k = \tau'_k$ for all k .*

Proof. For $n = 0$ or $n = 1$ there is nothing to prove. Suppose $\tau_1 \cup \dots \cup \tau_n = \tau'_1 \cup \dots \cup \tau'_n = (s_1, \dots, s_\ell)$ with $\tau_n = (s_k, \dots, s_\ell), \tau'_n = (s_{k'}, \dots, s_\ell) \in \mathbb{F}_{t_n}$. We have $s_k \cdots s_\ell = s_{k'} \cdots s_\ell$. Suppose $k < k'$. Since \mathbb{S} is cancellative, this implies $s_k \cdots s_{k'-1} = e$ and thus s_k is invertible which contradicts the fact that \mathbb{S} is conical. So $k \geq k'$. Analogously, we get $k' \geq k$ which shows $k = k'$ and thus $\tau_n = \tau'_n$. Now the proposition follows by induction. \square

4.6 Definition. A cancellative monoid \mathbb{S} is called a *unique factorization monoid* or *uf-monoid* for short if any two factorizations of the same element have a common refinement. A *cuf-monoid* is a conical uf-monoid.

Equivalently, a uf-monoid is a cancellative monoid such that \mathbb{F}_t is a directed set with respect to refinement for every $t \in \mathbb{S}$. The term *unique factorization monoid* or *uf-monoid* is used by Johnson [Joh71] for this kind of monoids. In the paper he gives different characterizations of uf-monoids and presents constructions to find uf-monoids. We only deal with cuf-monoids, because we will need Proposition 4.5. The examples we will use later on are only $\mathbb{N}_0, \mathbb{Q}_+$ and \mathbb{R}_+ , but it seems that cuf-monoids provide the most general setting in which we can study the inductive limit constructions of the following sections.

4.4 First Inductive Limit: The Generated Full Comonoidal System

Let $((A_s)_{s \in \mathbb{S}}, (\Delta_{s,t})_{s,t \in \mathbb{S}}, \delta)$ be a comonoidal system over a cancellative monoid \mathbb{S} in a tensor category (\mathcal{C}, \boxtimes) with unit object E . For a tuple $\sigma = (s_1, \dots, s_n) \in \mathbb{F}_t$ put $A_\sigma := A_{s_1} \boxtimes \dots \boxtimes A_{s_n}$ for $n \geq 1$ and $A_\emptyset := E$. Define $\Delta_\sigma: A_t \rightarrow A_\sigma$ recursively by

$$\begin{aligned} \Delta_\emptyset &:= \delta \\ \Delta_{(t)} &:= \text{id}_t \\ \Delta_{(s_1, \dots, s_{n+1})} &:= (\Delta_{(s_1, \dots, s_n)} \boxtimes \text{id}_{s_{n+1}}) \circ \Delta_{(s_1 \cdots s_n, s_{n+1})}. \end{aligned}$$

Let $\tau = (t_1, \dots, t_n) \in \mathbb{F}_t$ and $\sigma \geq \tau$. Since \mathbb{S} is cancellative, we can use Proposition 4.5 to write $\sigma = \tau_1 \cup \dots \cup \tau_n$ for uniquely determined $\tau_k \in \mathbb{F}_{t_k}$. With this notation we put

$$\Delta_\sigma^\tau := \Delta_{\tau_1} \boxtimes \dots \boxtimes \Delta_{\tau_n}: A_\tau \rightarrow A_\sigma$$

for all $\tau \leq \sigma \in \mathbb{F}_t$.

4.7 Lemma. *It holds that $\Delta_\rho^\sigma \circ \Delta_\sigma^\tau = \Delta_\rho^\tau$ for all $\tau \leq \sigma \leq \rho \in \mathbb{F}_t$ for every cancellative monoid.*

Proof. The proof of Bhat and Mukherjee for the case $\mathbb{S} = \mathbb{R}_+$ [BM10, Lemma 4] works without a change for general cancellative monoids. \square

4.8 Corollary. *Let \mathbb{S} be a cuf-monoid and $((A_s)_{s \in \mathbb{S}}, (\Delta_{s,t})_{s,t \in \mathbb{S}}, \delta)$ a comonoidal system in a tensor category \mathcal{C} . Then for every $t \in \mathbb{S}$, $((A_\tau)_{\tau \in \mathbb{F}_t}, (\Delta_\sigma^\tau)_{\sigma \geq \tau \in \mathbb{F}_t})$ is an inductive system.*

Proof. By Definition 4.6 of a uf-monoid, \mathbb{F}_t is directed. The first condition of an inductive system, $\Delta_\tau^\tau = \text{id}_\tau$, is obvious. The second condition, $\Delta_\rho^\sigma \circ \Delta_\sigma^\tau = \Delta_\rho^\tau$ for all $\tau \leq \sigma \leq \rho \in \mathbb{F}_t$, is the statement of Lemma 4.7. \square

Suppose that the inductive systems $(A_\tau)_{\tau \in \mathbb{F}_t}$ have inductive limits \mathcal{A}_t with morphisms $D^\tau: A_\tau \rightarrow \mathcal{A}_t$. For $\tau \in \mathbb{F}_t$ denote by \mathbb{F}_τ the set of all refinements of τ . Then \mathbb{F}_τ is a cofinal subset of \mathbb{F}_t , see Example 2.2. Denote by \mathcal{A}_τ the inductive limit. Then there is a canonical isomorphism $\mathcal{A}_t \cong \mathcal{A}_\tau$ because of Corollary 2.4.

4.9 Lemma. *The diagram*

$$\begin{array}{ccc} A_\sigma \boxtimes A_\tau & \xrightarrow{D^\sigma \boxtimes D^\tau} & \mathcal{A}_s \boxtimes \mathcal{A}_t \\ \Delta_{\sigma'}^\sigma \boxtimes \Delta_{\tau'}^\tau \downarrow & \nearrow D^{\sigma'} \boxtimes D^{\tau'} & \\ A_{\sigma'} \boxtimes A_{\tau'} & & \end{array}$$

commutes for all $\sigma' \geq \sigma \in \mathbb{F}_s, \tau' \geq \tau \in \mathbb{F}_t$.

Proof. By functoriality of \boxtimes , we have

$$(D^{\sigma'} \boxtimes D^{\tau'}) \circ (\Delta_{\sigma'}^\sigma \boxtimes \Delta_{\tau'}^\tau) = (D^{\sigma'} \circ \Delta_{\sigma'}^\sigma) \boxtimes (D^{\tau'} \circ \Delta_{\tau'}^\tau) = D^\sigma \boxtimes D^\tau. \quad \square$$

So, by the universal property of the inductive limit, there are unique morphisms $\tilde{\Delta}_{s,t}: \mathcal{A}_{st} \rightarrow \mathcal{A}_s \boxtimes \mathcal{A}_t$ such that

$$\begin{array}{ccc} A_\sigma \boxtimes A_\tau & \xrightarrow{D^\sigma \boxtimes D^\tau} & \mathcal{A}_s \boxtimes \mathcal{A}_t \\ \downarrow D^{\sigma \sim \tau} & \nearrow \tilde{\Delta}_{s,t} & \\ \mathcal{A}_{st} \cong \mathcal{A}_{(s,t)} & & \end{array}$$

commutes for every $\sigma \in \mathbb{F}_s, \tau \in \mathbb{F}_t$.

4.10 Theorem. *The \mathcal{A}_t form a comonoidal system with respect to the coproduct morphisms $\tilde{\Delta}_{s,t}$ and the counit morphism id_E .*

Proof. First note that $\mathcal{A}_e = E$, since $\mathbb{F}_e = \{()\}$ and $A_0 = E$. The counit property is trivially fulfilled. In the diagram

$$\begin{array}{ccccc}
 & & \mathcal{A}_{rst} & & \\
 & \swarrow \tilde{\Delta} & \uparrow D^\rho \sim \sigma \sim \tau & \searrow \tilde{\Delta} & \\
 \mathcal{A}_r \boxtimes \mathcal{A}_{st} & \xleftarrow{D^\rho \boxtimes D^\sigma \sim \tau} & A_\rho \boxtimes A_\sigma \boxtimes A_\tau & \xrightarrow{D^\rho \sim \sigma \boxtimes D^\tau} & \mathcal{A}_{rs} \boxtimes \mathcal{A}_t \\
 & \searrow \text{id} \boxtimes \tilde{\Delta} & \downarrow D^\rho \boxtimes D^\sigma \boxtimes D^\tau & \swarrow \tilde{\Delta} \boxtimes \text{id} & \\
 & & \mathcal{A}_r \boxtimes \mathcal{A}_s \boxtimes \mathcal{A}_t & &
 \end{array}$$

the four corners commute by the definition of $\tilde{\Delta}$ and this proves coassociativity. \square

Define $D_t: A_t \rightarrow \mathcal{A}_t$ by $D_t := D^{(t)}$ for $t \neq e$ and $D_e := \delta$.

4.11 Theorem. *The morphisms $(D_t)_{t \in \mathbb{S}}$ form a morphism of comonoidal systems, that is $\tilde{\Delta}_{s,t} \circ D_{st} = (D_s \boxtimes D_t) \circ \Delta_{s,t}$ and $\text{id}_E \circ D_e = \delta$.*

Proof. The counit is respected by definition of D_e . In the diagram

$$\begin{array}{ccc}
 A_{st} & \xrightarrow{\Delta_{s,t}} & A_s \boxtimes A_t \\
 \downarrow D^{(st)} & \swarrow D^{(s,t)} & \downarrow D^{(s)} \boxtimes D^{(t)} \\
 \mathcal{A}_{st} \cong \mathcal{A}_{(s,t)} & \xrightarrow{\tilde{\Delta}_{s,t}} & \mathcal{A}_s \boxtimes \mathcal{A}_t
 \end{array}$$

the lower right commutes by definition of $\tilde{\Delta}$ and the upper left because \mathcal{A}_{st} is the inductive limit. So the outside square commutes, which finishes the proof. \square

Let $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then any inductive system $((A_\alpha)_\alpha, (f_\beta^\alpha)_{\alpha \leq \beta})$ in \mathcal{C} is mapped to an inductive system $((\mathcal{F}(A_\alpha))_\alpha, (\mathcal{F}(f_\beta^\alpha))_{\alpha \leq \beta})$ in \mathcal{D} . We say that \mathcal{F} *preserves inductive limits* if for every inductive limit $(\mathcal{A}, (f_\alpha)_\alpha)$ of an inductive system $((A_\alpha)_\alpha, (f_\beta^\alpha)_{\alpha \leq \beta})$ it holds that $(\mathcal{F}(\mathcal{A}), (\mathcal{F}(f_\alpha))_\alpha)$ is an inductive limit of the inductive system $((\mathcal{F}(A_\alpha))_\alpha, (\mathcal{F}(f_\beta^\alpha))_{\alpha \leq \beta})$.

4.12 Theorem. *If the tensor product preserves inductive limits, the morphisms $\tilde{\Delta}_{s,t}$ are all isomorphisms. In other words $(\mathcal{A}_t)_{t \in \mathbb{S}}$ is a full comonoidal system.*

Proof. The tensor product is a bifunctor $\boxtimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$. Inductive systems in $\mathcal{C} \times \mathcal{C}$ are in bijection with pairs of inductive systems in \mathcal{C} and an inductive limit in $\mathcal{C} \times \mathcal{C}$ is a pair of inductive limits for the inductive systems in \mathcal{C} . If \boxtimes preserves inductive limits, $\mathcal{A}_s \boxtimes \mathcal{A}_t$ is an inductive limit of the inductive system formed by $(A_\sigma \boxtimes A_\tau)_{\sigma \in \mathbb{F}_s, \tau \in \mathbb{F}_t}$ with respect to the

maps $D^\sigma \boxtimes D^\tau$. Since $\tilde{\Delta}_{s,t}$ makes the diagram

$$\begin{array}{ccc} A_\sigma \boxtimes A_\tau & \xrightarrow{D^\sigma \boxtimes D^\tau} & \mathcal{A}_s \boxtimes \mathcal{A}_t \\ D^\sigma \sim \tau \downarrow & \nearrow \tilde{\Delta}_{s,t} & \\ \mathcal{A}_{st} \cong \mathcal{A}_{(s,t)} & & \end{array}$$

commute, it is the canonical isomorphism between the two inductive limits. \square

All tensor categories we are interested in have tensor products which do preserve inductive limits.

4.5 Ore Monoids

For Ore monoids there is a left and a right version, we only treat the right version.

4.13 Definition. Let \mathbb{S} be a monoid. For $s, r \in \mathbb{S}$ we write $s \leq r$ if s is a left divisor of r , that is if there exists a $p \in \mathbb{S}$ such that $sp = r$.

On every monoid the defined binary relation \leq is a preorder, that is it is reflexive and transitive.

4.14 Definition. A cancellative monoid \mathbb{S} is called an *Ore monoid* if for all $s, t \in \mathbb{S}$ there exists an $r \in \mathbb{S}$ with $s \leq r$ and $t \leq r$, i.e., if (\mathbb{S}, \leq) is directed.

Product systems of C^* -correspondences (Hilbert bimodules) over Ore monoids have been studied independently by Albandik and Meyer [AM15] and Kwaśniewski and Szymański [KS16]. Albandik and Meyer's definitions are slightly more general, because they allow also noncancellative monoids. This works as well, but one has to replace inductive limit by filtered colimits which many readers are probably less familiar with.

4.15 Theorem. Let \mathbb{S} be a cancellative monoid. Then the following are equivalent:

- \mathbb{S} is a *cuf-monoid* and an *Ore monoid*.
- \mathbb{S} is *totally ordered* with respect to \leq .

Proof. Suppose that bS is a cuf-monoid and an Ore monoid. Because \mathbb{S} is an Ore monoid, (\mathbb{S}, \leq) is a directed set. Let $s, t \in \mathbb{S}$. By the Ore property there exist $r, p, q \in \mathbb{S}$ with $r = sp = tq$. We want to show that $s \in t\mathbb{S}$ or $t \in s\mathbb{S}$. If one of the four elements s, t, p, q is e this is obvious. If none of them is invertible, then $(s, p), (t, q) \in \mathbb{F}_r$, so they have a common refinement $(r_1, \dots, r_n) \in \mathbb{F}_r$ because \mathbb{S} is a uf-monoid. This means $s = r_1 \cdots r_i, q = r_{i+1} \cdots r_n$ and $t = r_1 \cdots r_j, q = r_{j+1} \cdots r_n$ for some $i, j \in \{1, \dots, n\}$. Now $i \leq j$ implies $t = sr_{i+1} \cdots r_j \in s\mathbb{S}$ and $j \leq i$ implies $s = tr_{j+1} \cdots r_i \in t\mathbb{S}$, so we are done.

Now suppose that \mathbb{S} is totally ordered. Clearly, \mathbb{S} is an Ore monoid. Let $\varepsilon \in U(\mathbb{S})$. Then $\varepsilon \leq e$ and $e \leq \varepsilon$ imply $\varepsilon = e$, so there are nontrivial invertible elements. We prove that for all $r \in \mathbb{S}$ and all $(t_1, \dots, t_n), (s_1, \dots, s_m) \in \mathbb{F}_r$ there is a common refinement by induction on n . For $n = 0$ and $n = 1$ this is obvious. Now suppose the statement holds for $n - 1 \in \mathbb{N}$ and let $(t_1, \dots, t_n), (s_1, \dots, s_m) \in \mathbb{F}_r$. Because $t_1 \cdots t_n = r$ we know that $t_1 \cdots t_{n-1} < r$. Because \leq is a total order, we are in one of the following situations:

Case 1 $t_1 \cdots t_{n-1} = s_1 \cdots s_k$ for some $k \in \{1, \dots, m - 1\}$. By the induction hypothesis, there is a common refinement (r_1, \dots, r_ℓ) of (t_1, \dots, t_{n-1}) and (s_1, \dots, s_k) . It is easy to check that $(r_1, \dots, r_\ell, s_{k+1}, \dots, s_m)$ is a common refinement of (t_1, \dots, t_n) and (s_1, \dots, s_m) .

Case 2: $s_1 \cdots s_k < t_1 \cdots t_{n-1} < s_1 \cdots s_{k+1}$ for some $k \in \{1, \dots, m - 1\}$. Then there are p, q such that $s_1 \cdots s_k p = t_1 \cdots t_{n-1}$ and $t_1 \cdots t_{n-1} q = s_1 \cdots s_{k+1}$. By the induction hypothesis, there exists a common refinement (r_1, \dots, r_ℓ) of (s_1, \dots, s_k, p) and (t_1, \dots, t_{n-1}) . Now it follows that $(r_1, \dots, r_\ell, q, s_{k+2}, \dots, s_m)$ is a common refinement of (t_1, \dots, t_n) and (s_1, \dots, s_m) . \square

4.6 Second Inductive Limit: Lévy-Processes

Let \mathcal{A}_t be a full comonoidal system over an Ore monoid \mathbb{S} with coproduct isomorphisms $\tilde{\Delta}_{s,t}$ in a tensor category initial unit object. Without loss of generality assume that $\mathcal{A}_e = E$ and that the counit morphism is $\delta = \text{id}_E$.

For $s \leq t, t = sp$, define $i_t^s: \mathcal{A}_s \rightarrow \mathcal{A}_t$ as the composition

$$\mathcal{A}_s \xrightarrow{\iota^1} \mathcal{A}_s \boxtimes \mathcal{A}_p \xrightarrow{\tilde{\Delta}^{-1}} \mathcal{A}_t.$$

4.16 Theorem. $((\mathcal{A}_t)_{t \in \mathbb{S}}, (i_t^s)_{s \leq t})$ is an inductive system.

Proof. Let $r \leq s \leq t, s = rp, t = sq$. In the diagram

$$\begin{array}{ccccc} \mathcal{A}_r & \xrightarrow{\iota^1} & \mathcal{A}_r \boxtimes \mathcal{A}_p & \xrightarrow{\tilde{\Delta}^{-1}} & \mathcal{A}_s \\ \downarrow \text{id} & & \downarrow \text{id} \boxtimes \iota^1 & & \downarrow \iota^1 \\ \mathcal{A}_r & \xrightarrow{\iota^1} & \mathcal{A}_r \boxtimes \mathcal{A}_p \boxtimes \mathcal{A}_q & \xrightarrow{\tilde{\Delta}^{-1} \boxtimes \text{id}} & \mathcal{A}_s \boxtimes \mathcal{A}_q \\ \downarrow \text{id} & & \downarrow \text{id} \boxtimes \tilde{\Delta}^{-1} & & \downarrow \tilde{\Delta}^{-1} \\ \mathcal{A}_r & \xrightarrow{\iota^1} & \mathcal{A}_r \boxtimes \mathcal{A}_{pq} & \xrightarrow{\tilde{\Delta}^{-1}} & \mathcal{A}_t \end{array}$$

the lower right corner commutes by coassociativity of $\tilde{\Delta}$ and the other three corners commute by the naturality of ι^1 . We suppressed the associativity constraint and identified $\mathcal{A}_r \boxtimes (\mathcal{A}_p \boxtimes \mathcal{A}_q)$ with $(\mathcal{A}_r \boxtimes \mathcal{A}_p) \boxtimes \mathcal{A}_q$, which leads to the two interpretations $\text{id}_{\mathcal{A}_r} \boxtimes \iota_{\mathcal{A}_p, \mathcal{A}_q}^1$ and $\iota_{\mathcal{A}_r \boxtimes \mathcal{A}_p, \mathcal{A}_q}^1$.

of the arrow $\mathcal{A}_r \boxtimes \mathcal{A}_p \rightarrow \mathcal{A}_r \boxtimes \mathcal{A}_p \boxtimes \mathcal{A}_q$. Of course, this would not work without coherence as discussed in Observation 3.6. \square

For the rest of this section, we fix a totally ordered monoid \mathbb{S} and an inductively complete tensor category \mathcal{C} with compatible inclusions ι^1, ι^2 , whose tensor product \boxtimes preserves inductive limits.

4.17 Remark. As the typical examples of totally ordered monoids are submonoids of \mathbb{R}_+ , we will use additive notation in the following. For $s \leq t$, we denote the unique element r with $s + r = t$ by $t - s$.

Given only the comonoidal system $((\mathcal{A}_t)_{t \in \mathbb{S}}, (\Delta_{s,t})_{s,t \in \mathbb{S}})$ one can construct a canonical categorical Lévy-process. Since \mathbb{S} is a uf-monoid and \boxtimes preserves inductive limits, $(\mathcal{A}_t)_{t \in \mathbb{S}}$ generates a full comonoidal system $((\mathcal{A}_t)_{t \in \mathbb{S}}, \tilde{\Delta})$ by Theorem 4.12. Denote by $D^t: \mathcal{A}_t \rightarrow \mathcal{A}_t$ the canonical morphisms. Let $(\mathcal{A}, (i^t: \mathcal{A}_t \rightarrow \mathcal{A})_{t \in \mathbb{S}})$ the inductive limit of $(\mathcal{A}_t)_{t \in \mathbb{S}}$. Define $j_{s,t}: \mathcal{A}_{t-s} \rightarrow \mathcal{A}$ as the composition

$$\mathcal{A}_{t-s} \xrightarrow{D^{t-s}} \mathcal{A}_{t-s} \xrightarrow{\iota^2} \mathcal{A}_s \boxtimes \mathcal{A}_{t-s} \xrightarrow{\tilde{\Delta}^{-1}} \mathcal{A}_t \xrightarrow{i^t} \mathcal{A}.$$

4.18 Theorem. *The $j_{s,t}$ form a categorical Lévy process.*

Proof. We construct the independence morphism $j_{r,s,t}$ for $j_{r,s}$ and $j_{s,t}$ and show that $j_{r,s,t} \circ \Delta_{s-r,t-s} = j_{r,t}$. Define $j_{r,s,t}$ as the composition

$$\mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} \xrightarrow{D^{s-r} \boxtimes D^{t-s}} \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} \xrightarrow{\iota^2} \mathcal{A}_r \boxtimes \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} \xrightarrow{\tilde{\Delta}^{-1}} \mathcal{A}_t \xrightarrow{i^t} \mathcal{A}.$$

Now, the diagram

$$\begin{array}{ccccccc} \mathcal{A}_{t-s} & \xrightarrow{D^{t-s}} & \mathcal{A}_{t-s} & \xrightarrow{\iota^2} & \mathcal{A}_s \boxtimes \mathcal{A}_{t-s} & \xrightarrow{\tilde{\Delta}^{-1}} & \mathcal{A}_t \\ \downarrow \iota^2 & & \downarrow \iota^2 & & \downarrow \tilde{\Delta} & & \downarrow \text{id} \\ \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} & \xrightarrow{D^{s-r} \boxtimes D^{t-s}} & \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} & \xrightarrow{\iota^2} & \mathcal{A}_r \boxtimes \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} & \xrightarrow{\tilde{\Delta}^{-1}} & \mathcal{A}_t \\ \uparrow \iota^1 & & \uparrow \iota^1 & & \uparrow \iota^1 & & \uparrow i_t^s \\ \mathcal{A}_{s-r} & \xrightarrow{D^{s-r}} & \mathcal{A}_{s-r} & \xrightarrow{\iota^2} & \mathcal{A}_r \boxtimes \mathcal{A}_{s-r} & \xrightarrow{\tilde{\Delta}^{-1}} & \mathcal{A}_s \end{array}$$

$\nearrow i^t$
 $\nwarrow i^s$

commutes: The leftmost squares commute due to naturality of ι^1 and ι^2 . The next squares commute by Observation 3.6 and naturality of ι^2 . The upper right square commutes by coassociativity of $\tilde{\Delta}$. The triangles commute by definition of the inductive limit. It remains

to show commutativity of the lower right square. In a bit more detail, this is

$$\begin{array}{ccccc}
 \mathcal{A}_r \boxtimes \mathcal{A}_{s-r} \boxtimes \mathcal{A}_{t-s} & \xrightarrow{\tilde{\Delta}^{-1} \boxtimes \text{id}} & \mathcal{A}_s \boxtimes \mathcal{A}_{t-s} & \xrightarrow{\tilde{\Delta}} & \mathcal{A}_t \\
 \uparrow \iota^1 & & \uparrow \iota^1 & \nearrow i_t^s & \\
 \mathcal{A}_r \boxtimes \mathcal{A}_{s-r} & \xrightarrow{\tilde{\Delta}^{-1}} & \mathcal{A}_s & &
 \end{array}$$

which commutes by naturality of ι^1 and the definition of i_t^s . This shows that $j_{r,s,t}$ is an independence morphism. Next we consider

$$\begin{array}{ccccccc}
 A_{t-r} & \xrightarrow{D^{t-r}} & A_{t-r} & \xrightarrow{\iota^2} & \mathcal{A}_r \boxtimes \mathcal{A}_{t-r} & \xrightarrow{\tilde{\Delta}^{-1}} & \mathcal{A}_t \text{id} \\
 \downarrow \Delta_{s-r,t-s} & & \downarrow \tilde{\Delta} & & \downarrow \text{id} \boxtimes \tilde{\Delta} & & \downarrow i^t \\
 A_{s-r} \boxtimes A_{t-s} & \xrightarrow{D^{s-r} \boxtimes D^{t-s}} & A_{s-r} \boxtimes A_{t-s} & \xrightarrow{\iota^2} & \mathcal{A}_r \boxtimes A_{s-r} \boxtimes A_{t-s} & \xrightarrow{\tilde{\Delta}^{-1}} & \mathcal{A}_t \xrightarrow{i^t} \mathcal{A}
 \end{array}$$

in which the first square commutes because the D^t form a morphism of comonoidal systems by Theorem 4.11, the second square commutes due to naturality of ι^2 , and the last square and the triangle commute trivially. So the outside commutes, thus establishing $j_{r,s,t} \circ \Delta_{s-r,t-s} = j_{r,t}$.

The general construction of an independence morphism for $j_{t_1,t_2}, \dots, j_{t_n,t_{n+1}}$ works similar to that for $j_{r,s}$ and $j_{s,t}$. \square

There is also a direct way from the comonoidal system $(A_t)_{t \in \mathbb{S}}$ to the Lévy process. Put $\mathbb{F} := \{\sigma = (s_1, \dots, s_n) \mid n \in \mathbb{N}_0, s_k \in \mathbb{S} \setminus \{e\}\} = \bigcup_{s \in \mathbb{S}} \mathbb{F}_s$ and define $\sigma \geq \tau = (t_1, \dots, t_n)$ if there exist $\tau_1, \dots, \tau_n, \tau_{n+1}$ with $\sigma = \tau_1 \smile \dots \smile \tau_n \smile \tau_{n+1}$, $\tau_k \in \mathbb{F}_{t_k}$ for $k \in \{1, \dots, n\}$ and $\tau_{n+1} \in \mathbb{F}$. One shows that \mathbb{F} is directed analogously to \mathbb{F}_t . Then define an inductive system $(A_\sigma)_{\sigma \in \mathbb{F}}$ with respect to the morphisms $i_\sigma^\tau: A_\tau \rightarrow A_\sigma$ defined as the composition

$$A_\tau \xrightarrow{\Delta_{\tau_1 \smile \dots \smile \tau_n}^\tau} A_{\tau_1} \boxtimes \dots \boxtimes A_{\tau_n} \xrightarrow{\iota^1} A_{\tau_1} \boxtimes \dots \boxtimes A_{\tau_n} \boxtimes A_{\tau_{n+1}} = A_\sigma$$

for $\sigma = \tau_1 \smile \dots \smile \tau_n \smile \tau_{n+1} \geq \tau$.

4.19 Theorem. *The inductive limits of $(A_\sigma)_{\sigma \in \mathbb{F}}$ and $(\mathcal{A}_t)_{t \in \mathbb{S}}$ are canonically isomorphic.*

Proof. This is exactly the situation of Proposition 2.3. \square

5 Examples

We explore what categorial independence means in several concrete tensor categories. First we shall look at some trivial examples, in order to see that categorial independence encompasses notions such as linear independence and orthogonality. Then we shall see that the

quantum probabilistic notions of independence induced by universal products (e.g. Voiculescu's freeness) and newer generalizations are covered and yield quantum Lévy processes.

5.1 Example. Independence in quantum probability is usually implemented by a *universal product*, which is a prescription \square that assigns to two linear functionals on algebras $\mathcal{A}_1, \mathcal{A}_2$ a new linear functional $\varphi_1 \square \varphi_2$ on the *free product* $\mathcal{A}_1 \sqcup \mathcal{A}_2$ such that the bifunctor $((\mathcal{A}_1, \Phi_1), (\mathcal{A}_2, \Phi_2)) \mapsto (\mathcal{A}_1 \sqcup \mathcal{A}_2, \Phi_1 \square \Phi_2)$ turns the category $\mathbf{alg}\Omega$ of quantum probability spaces into a tensor category with the canonical embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}_1 \sqcup \mathcal{A}_2$ as inclusions. An example is the tensor product of linear functionals defined by

$$\Phi_1 \otimes \Phi_2(a_1 \cdots a_n) = \Phi_1 \left(\prod_{a_i \in \mathcal{A}_1}^{\rightarrow} a_i \right) \Phi_2 \left(\prod_{a_i \in \mathcal{A}_2}^{\rightarrow} a_i \right),$$

where \prod^{\rightarrow} denotes the product of the algebra elements in the same order as they appear in $a_1 \cdots a_n$. In this case categorial independence reproduces the notion of *tensor-independence*. If \square is the *free product*, we get *freeness*.

Although Definition 3.2 was motivated by quantum probability, it encompasses also non-stochastic notions of independence, as the following examples show.

5.2 Example (Linear Independence). Consider the category \mathbf{vec}^{inj} of vector spaces with injective linear maps. The direct sum turns this into a tensor category with inclusions with respect to the canonical embeddings $V_i \hookrightarrow V_1 \oplus V_2$. Two injections $f_i: V_i \rightarrow W$ are independent if and only if they have linearly independent ranges. The only choice for the independence morphism is the linear map $h := f_1 + f_2: V_1 \oplus V_2 \rightarrow W$. If h is injective, then $f_1(v_1) + f_2(v_2) = 0$ implies $f_1(v_1) = f_2(v_2) = 0$, so the ranges are linearly independent. On the other hand, if the ranges are linearly independent and $h(v_1 \oplus v_2) = 0$, we can conclude that $f_i(v_i) = 0$ for $i \in \{1, 2\}$. Since f_1 and f_2 are injections, it follows that $v_1 = 0$ and $v_2 = 0$, so h is injective.

5.3 Example (Orthogonality). Similar to the previous example, $(\mathbf{hilb}^{isom}, \oplus)$ is a tensor category with the canonical embeddings as inclusions. Two isometries $v_i: H_i \rightarrow G$ are independent if and only if they have orthogonal ranges. Indeed, the only choice for the independence morphism is the linear map $h = v_1 + v_2$. This is an isometry if and only if

$$\begin{aligned} 0 &= \langle v_1(x_1) + v_2(x_2), v_1(x_1) + v_2(x_2) \rangle - \langle x_1 \oplus x_2, x_1 \oplus x_2 \rangle \\ &= \langle v_1(x_1), v_2(x_2) \rangle + \langle v_2(x_2), v_1(x_1) \rangle \\ &= 2\Re \langle v_1(x_1), v_2(x_2) \rangle \end{aligned}$$

for all $x_1 \in H_1, x_2 \in H_2$, which is clearly equivalent to $v_1(H_1) \perp v_2(H_2)$.

5.1 Nonprobabilistic Categories

Sets Consider the tensor category $(\mathbf{set}^{inj}, \times)$ with the unit object E . Note that E is necessarily a one point set $E = \{\Lambda\}$. If $(A_s)_{s \in \mathbb{S}}$ is a comonoidal system, injectivity of the counit morphism $\delta: A_e \rightarrow E$ implies that A_e is either empty, or also a one point set. A comonoidal system $(A_s)_{s \in \mathbb{S}}$ with $A_e = \{\Lambda\}$ is called a *Cartesian system*. Cartesian systems over $\mathbb{N}_0, \mathbb{Q}_+$ and \mathbb{R}_+ appear a lot in [GS14]. The unit object is a one point set, which is not an initial object in \mathbf{set} . So there at a first glance there is no notion of independence here. However, we can also consider the same category with the disjoint union instead of Cartesian product as tensor product. In this case the unit object is the empty set. It is easy to see that injections $f_i: B_i \rightarrow A$ are independent if and only if the $f_i(B_i)$ are pairwise disjoint.

Vector Spaces Monoids in (\mathbf{vec}, \otimes) are unital algebras, comonoids are coalgebras. Let $(A_t)_{t \in \mathbb{S}}$ be a monoidal system in (\mathbf{vec}, \otimes) . Then $A := \bigoplus_{t \in \mathbb{S}} A_t$ is an \mathbb{S} -graded algebra with respect to the multiplication given by

$$ab := \mu_{s,t}(a \otimes b)$$

for elements $a \in A_s, b \in A_t$. If we consider $(\mathbf{vec}^{surj}, \otimes)$, a monoidal system over \mathbb{N}_0 yields a *standard graded algebra*, that is an \mathbb{N}_0 -graded algebra $A = \bigoplus_{n \in \mathbb{N}_0} A_n$ with $A_0 = \mathbb{C}1$ and $A_m A_n = A_{m+n}$. Indeed, the two conditions are exactly the surjectivity of the unit morphism $1: \mathbb{C} \rightarrow A_0$ and the product morphisms $\mu_{m,n}: A_m \otimes A_n \rightarrow A_{m+n}$.

Hilbert Spaces Comonoidal systems in $(\mathbf{hiltb}^{isom}, \otimes)$ with $H_e = \mathbb{C}$ and $\delta = \text{id}_{\mathbb{C}}$ are called *subproduct systems*. Subproduct systems over $\mathbb{N}_0, \mathbb{Q}_+$ and \mathbb{R}_+ are the main subjects of [GS14]. Full subproduct systems are called *product systems*. Actually, defining subproduct systems as monoidal systems in $(\mathbf{hiltb}^{coisom}, \otimes)$ gives an equivalent definition. More precisely $((H_s)_{s \in \mathbb{S}}, (\Delta_{s,t})_{s,t \in \mathbb{S}}, \delta)$ is a comonoidal system in $(\mathbf{hiltb}^{isom}, \otimes)$ if and only if $((H_s)_{s \in \mathbb{S}}, (\Delta_{s,t}^*)_{s,t \in \mathbb{S}}, \delta^*)$ is a monoidal system in $(\mathbf{hiltb}^{coisom}, \otimes)$.

5.4 Remark. The forgetful functor $\mathcal{F}: (\mathbf{finhiltb}^{coisom}, \otimes) \rightarrow (\mathbf{finvec}^{surj})$ is easily seen to be a tensor functor. Tensor functors map monoidal systems to monoidal systems (just as cotensor functors do with comonoidal systems). So it follows from the previous paragraph that $(\mathcal{F}(H_n))_{n \in \mathbb{N}_0}$ yields a standard graded algebra if $(H_n)_{n \in \mathbb{N}_0}$ is a subproduct system.

Algebras Comonoids in $(\mathbf{alg}_1, \otimes)$ are called *bialgebras*, those in $*\text{-}\mathbf{alg}_1$ are referred to as **-bialgebras*. Comonoids in (\mathbf{alg}, \sqcup) are called *dual semigroups*. The additive deformations of [Wir02, Ger11, GKL12] provide examples of comonoidal systems in $(\mathbf{alg}_1, \otimes)$ and $(*\text{-}\mathbf{alg}_1, \otimes)$.

5.2 Probability Spaces

Denote by \mathbf{prob} the category with probability spaces as objects and measurable maps $f: \Omega_1 \rightarrow \Omega_2$ with $\mathbb{P}_1 \circ f^{-1} = \mathbb{P}_2$ as morphisms from $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ to $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. This is a tensor category with $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1) \otimes (\Omega_2, \mathcal{F}_2, \mathbb{P}_2) := (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$, where $\mathcal{F}_1 \otimes \mathcal{F}_2$ and $\mathbb{P}_1 \otimes \mathbb{P}_2$ are the product σ -algebra and product measure respectively. The unit object is a one-point probability space Λ . Now Λ is clearly terminal, not final. To use our definitions of independence and Lévy processes we could either restate everything for the situation of the unit object being terminal, in which case we would have to reverse all arrows, or we can simply switch to the opposite category \mathbf{prob}^{op} . Both ways give basically the same definition of independence and, as noted by Franz [Fra06], this categorial independence coincides with the usual notion of stochastic independence in the following way: Let $X: E \rightarrow \Omega, Y: F \rightarrow \Omega$ be two random variables. Pulling back the probability measure on Ω to E and F turns these into probability spaces themselves and X and Y can be interpreted as morphisms in \mathbf{prob} (or \mathbf{prob}^{op}). Now X and Y are stochastically independent as random variables if and only if they are categorially independent as morphisms.

Now let $(\mu_t)_{t \geq 0}$ be a convolution semigroup of probability measures on the real line. In \mathbf{prob}^{op} the addition is a morphism from \mathbb{R} to $\mathbb{R} \times \mathbb{R}$ and the probability spaces $A_t := (\mathbb{R}, \mathcal{B}, \mu_t)$ form a comonoidal system. Indeed, $\mu_s \star \mu_t = \mu_{s+t}$ can be expressed by saying that addition transports the product measure $\mu_s \otimes \mu_t$ on $\mathbb{R} \times \mathbb{R}$ to the measure μ_{s+t} on \mathbb{R} , so addition can be interpreted as a morphism from A_{s+r} to $A_s \otimes A_t$ in \mathbf{prob}^{op} . The inductive limits discussed in Theorem 4.19 are the same as the projective limit from the Daniell-Kolmogoroff theorem (taken in \mathbf{prob}).

5.3 Tensor Product of Algebraic Quantum Probability Spaces

In the widest sense, an algebraic quantum probability space is a pair (A, φ) which consists of an algebra A (generalizing the algebra of bounded measurable functions on a probability space) and linear functional $\varphi: A \rightarrow \mathbb{C}$ (generalizing the expectation/integral with respect to a probability measure). The category \mathbf{algQ} is formed by all algebraic quantum probability spaces as objects and algebra homomorphisms $j: A_1 \rightarrow A_2$ with $\varphi_2 \circ j = \varphi_1$ as morphism from (A_1, φ_1) to (A_2, φ_2) . By \mathbf{algQ}_1 we denote the subcategory whose objects consist of a unital algebra with a unital linear functional and whose morphisms are only those morphisms of \mathbf{algQ} which are unital algebra homomorphisms.

Maybe the simplest tensor category in this context is $(\mathbf{algQ}_1, \otimes)$, with tensor product $(A_1, \varphi_1) \otimes (A_2, \varphi_2) := (A_1 \otimes A_2, \varphi_1 \otimes \varphi_2)$ and unit object $(\mathbb{C}, \text{id}_{\mathbb{C}})$. The unit object is initial in \mathbf{algQ}_1 , so we have a derived notion of independence for morphisms. Two morphisms $j_i: (A_i, \varphi_i) \rightarrow (A, \Phi)$, $i \in \{1, 2\}$, are independent if and only if the images of j_1 and j_2 commute with each other and $\Phi \circ \mu_A \circ (j_1 \otimes j_2) = (\Phi \circ j_1) \otimes (\Phi \circ j_2)$. Indeed, if there exists an independence morphism $h: (A_1, \varphi_1) \otimes (A_2, \varphi_2) \rightarrow (A, \Phi)$, then it must hold that

$$h(a \otimes b) = h(a \otimes 1)h(1 \otimes b) = j_1(a)j_2(b) = \mu_A \circ (j_1 \otimes j_2)(a \otimes b)$$

and

$$\Phi \circ \mu_A \circ (j_1 \otimes j_2) = \Phi \circ h = \varphi_1 \otimes \varphi_2 = (\Phi \circ j_1) \otimes (\Phi \circ j_2).$$

The fact that h is an algebra homomorphism implies

$$j_1(a)j_2(b) = \mu_A \circ (j_1 \otimes j_2)(1 \otimes b \cdot a \otimes 1) = j_2(b)j_1(a)$$

so the images commute. On the other hand, if $\Phi \circ \mu_A \circ (j_1 \otimes j_2) = (\Phi \circ j_1) \otimes (\Phi \circ j_2)$, then $h := \mu_A \circ (j_1 \otimes j_2)$ respects the linear functionals, and if the images commute, it is an algebra homomorphism, hence a (unique) independence morphism.

Now let B be a bialgebra with comultiplication Δ and counit δ . Then linear functionals on B can be convolved. Every convolution semigroup $(\varphi_t)_{t \geq 0}$ now gives rise to a comonoidal system $A_t = (B, \varphi_t)$, because $\varphi_{s+t} = \varphi_s \star \varphi_t$ is equivalent to Δ being a morphism in $\mathbf{alg}\mathfrak{Q}_1$ from A_{s+t} to $A_s \otimes A_t$ and $\varphi_0 = \delta$ is equivalent to δ being a morphism from A_0 to $(\mathbb{C}, \text{id}_{\mathbb{C}})$. If one also requires pointwise continuity, the resulting Lévy processes are the quantum Lévy processes on bialgebras whose theory is developed in [Sch93].

The setting of comonoidal system gives us more freedom. It is not necessary that the algebras of the A_t are all the same. The situation comes up when one considers Lévy processes on additive deformations of bialgebras. An *additive deformation* of a bialgebra B is a family of multiplication maps $\mu_t: B \otimes B \rightarrow B$ such that μ_0 is the original bialgebra multiplication, $\delta \circ \mu_t$ is pointwise continuous, $B_t := (B, \mu_t, 1)$ is a unital algebra for every $t \geq 0$ and $(\mu_s \otimes \mu_t) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta \otimes \Delta) = \Delta \circ \mu_{s+t}$ for all $s, t \geq 0$, cf. [Wir02] or [Ger11]. The existence of nontrivial additive deformations is determined by the Hochschild cohomology of B . Again, a convolution semigroup of linear functionals φ_t on the algebras $B_t := (B, \mu_t)$ gives rise to a comonoidal system $A_t = (B_t, \varphi_t)$ and corresponding Lévy processes.

Important variations are to look at categories of unital \ast -algebras with states or work with braided categories.

5.4 Universal Products

In quantum probability there are five well known notions of independence which come from universal products.

In order to motivate universal products, suppose that there is any bifunctor \boxtimes which turns $(\mathbf{alg}\mathfrak{Q}, \boxtimes)$ into a tensor category such that the unit object is the initial object $\{0\}$. By Theorem 3.5(b), there are inclusions $Q_i \rightarrow Q_1 \boxtimes Q_2$. Denote by A_Q and φ_Q the algebra and the linear functional of Q , i.e. $Q = (A_Q, \varphi_Q)$. Then the inclusions are algebra homomorphisms $A_{Q_i} \rightarrow A_{Q_1 \boxtimes Q_2}$. Denote by $A_1 \sqcup A_2$ the free product of the algebras A_1 and A_2 . The universal property of the free product establishes an algebra homomorphism $\iota^1 \sqcup \iota^2: A_{Q_1} \sqcup A_{Q_2} \rightarrow A_{Q_1 \boxtimes Q_2}$. For linear functionals $\varphi_i: A_i \rightarrow \mathbb{C}$ put $Q_i := (A_i, \varphi_i)$ and define $\varphi_1 \odot \varphi_2: A_1 \sqcup A_2$ as $\varphi_1 \odot \varphi_2 := \varphi_{Q_1 \boxtimes Q_2} \circ (\iota^1 \sqcup \iota^2)$. Then \odot fulfills the axioms of a universal product (cf. [GL15, Definition 3.1]).

A dual semigroup is by definition a comonoid in the category \mathbf{alg} of algebras. With re-

spect to a fixed universal product \boxplus , a convolution product for linear functionals on a dual semigroup can be defined via $\varphi_1 \star \varphi_2 := (\varphi_1 \boxplus \varphi_2) \circ \Delta$, where $\Delta: D \rightarrow D \sqcup D$ is the comultiplication of the dual semigroup D . As in the tensor product case, convolution semigroups can be studied and lead comonoidal systems $A_t := (D, \varphi_t)$ and finally to Lévy processes whose increments are independent in the sense given by the universal product (for example free, monotone or Boolean independence).

Universal products in the category $\mathbf{alg}\mathfrak{Q}$ have been almost completely classified (see [Spe97, BGS02, Mur03, GL15]). Correspondingly, the theory of Lévy processes could be dealt with by studying the special cases [BGS05]. There are two important generalizations of universal products for which a classification is out of reach, and which thus make it much more valuable to have the general path from convolution semigroups to Lévy processes available just working with the axioms. First, Bozeiko, Leinert and Speicher studied so-called *c-freeness*, which is obtained by taking products of pairs of linear functionals on an algebra [BLS96]. Following this, Hasebe introduced the *indented product*, which is a product for triples of linear functionals [Has10] and studied also a product for infinitely many functionals [Has11]. So instead of $\mathbf{alg}\mathfrak{Q}$, these products give bifunctors for a category $\mathbf{alg}\mathfrak{Q}_n$ whose objects are tuples $(A, \varphi_1, \dots, \varphi_n)$. Recently, in a series of papers [Voi14, Voi16a, Voi16b] Voiculescu presented a fascinating new notion of independence which he calls *bifreeness*. For bifreeness algebras have to come with a free product decomposition $A = A_l \sqcup A_r$ where A_l contains the *left variables* and A_r the *right variables*. Although no examples are yet known, many of the theory which can be done for universal products and bifreeness can also be done when the algebra is assumed to be decomposed into a free product of any fixed finite number of subalgebras. One comes to study the category $\mathbf{alg}\mathfrak{Q}^m$ whose objects are tuples $(A, A_1, \dots, A_m, \varphi)$, $\varphi: A = A_1 \sqcup \dots \sqcup A_m \rightarrow \mathbb{C}$. It is even possible to combine the two generalizations and work in a category $\mathbf{alg}\mathfrak{Q}_n^m$, defined in the obvious way. For all of these categories one can define universal products, which we suggest to call *polyuniversal products*, and develop the theory of Lévy processes, as discussed in this paper, as well as other topics related to independence, as has been successfully done with cumulants in [MS16].

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